LITERATURE CITED

1. F. L. Sayakhov, S. N. Chistyakov, G. A. Babalyan, and B. N. Fedorov, "Calculating the heating of surface zones in petroleum wells by means of high-frequency electromagnetic fields," Izv. Vuzov. Neft' Gaz, No. 2 (1972).
2. F. L. Sayakhov, M. A. Fatykhov, and O. L. Kuznetsov, "Investigating the electromagnetic-acoustic effect on temperature distribution in petroleum and water-saturated rocks," Izv. Vuzov, Neft' Gaz, No. 3 (1981).
3. Zyung Ngok Khai, A. G. Kutushev, and R. I. Nigmatulin, "The theory of filtration for liquids in a porous medium with volumetric heating by means of high-frequency electromagnetic fields," Prikl. Mat. Mekh., 51, No. 1 (1987).
4. Zyunk Ngok Khai, N. D. Musaev, and R. I. Nigmatulin, "Self-similar solutions of problems in heat and mass transfer in a saturated porous medium with volumetric heat sources," Prikl. Mat. Mekh., 51, No. 6 (1987).
5. A. A. Samarskii, The Theory of Different Schemes [in Russian], Nauka, Moscow (1977).
6. V. M. Paskonov, V. I. Polezhaev, and L. A. Chudov, Numerical Modeling of Heat and Mass Exchange Processes [in Russian], Nauka, Moscow (1984).

AXISYMMETRIC FUNDAMENTAL SOLUTIONS FOR THE EQUATIONS OF HEAT CONDUCTION IN THE CASE OF CYLINDRICAL ANISOTROPY OF A MEDIUM

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UDC 517.95

Numerical methods of solving the boundary-value problems for the equations of mathematical physics based on the application of fundamental solutions, i.e., solutions describing the reaction of infinite space or an infinite plane to a concentrated action, are currently in widespread favor. Among these methods we can include the direct and indirect methods of boundary integral equations [1], as well as the method of sources in which the solution of the boundary-value problem is constructed by superposition of concentrated actions in space, above some surface encompassing the area under investigation [2]. For the equations of steady and nonsteady heat conduction in an isotropic medium such solutions are well established (see [1] and the references cited there) both for the two- and three-dimensional cases, as well as for the case of the axisymmetric problem. The plane and three-dimensional equations of heat conduction for a rectilinear anisotropic medium can be reduced to the isotropic case. We know of three-dimensional fundamental solutions for the equations of elasticity theory in the case of a medium with rectilinear anisotropy [3] and for a rectilinear anisotropic hereditary (or memory) elastic medium [4, 5].

The axisymmetric fundamental solutions for the steady and nonsteady equations of heat conduction in the case of a cylindrical anisotropic medium are constructed in the present study by reducing them to the corresponding equations for isotropic media. We present the limit relationships for the characteristic parametric values. As one of the limit cases we have derived the fundamental solutions for the steady and nonsteady equations of plane heat-conduction problems for a rectilinear anisotropic medium.

The equations of nonsteady heat conduction in an arbitrarily anisotropic medium have the form

$$\operatorname{div} \mathbf{q} + cT_{,t} = Q, \quad \mathbf{q} = -\chi\nabla T. \tag{0.1}$$

Here T is the temperature; $\mathbf{q}$ is the heat-flux vector; Q is the specified release of heat; c is the coefficient of heat capacity; χ is the symmetric heat-conduction tensor; t is time. The subscript which appears after the comma denotes the derivative with respect to the cor-

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responding coordinate; no summation is assumed over the repeated subscripts in the following.

Let the medium exhibit cylindrical anisotropy and let (r, θ, z) represent the cylindrical system of coordinates whose z axis coincides with the axis of anisotropy. Then, in this coordinate system the components of the heat-conduction tensor $\chi_{\alpha\beta}(\alpha, \beta = r, \theta, z)$ are constant and substitution of the second of the equations from (0.1) into the first yields

$$\chi_{rr}(T_{,rr} + T_{,r}/r) + \chi_{\theta\theta}T_{,\theta\theta}/r^2 + \chi_{zz}T_{,zz} + 2\chi_{r\theta}T_{,r\theta}/r + \\ + \chi_{rz}(2T_{,rz} + T_{,z}/r) + 2\chi_{z\theta}T_{,\theta z}/r - cT_{,t} = -Q(r, \theta, z, t). \tag{0.2}$$

1. The Axisymmetric Fundamental Solution of the Steady Equation. We will look for the solution of (0.2) with a right-hand side $Q = Q_0\delta(r - r_0)\delta(z - z_0)$ (δ is the Dirac delta function and Q_0 is a constant, with (r_0, z_0) representing the coordinate of the source). Since the right-hand side depends neither on θ nor t, this relationship disappears in (0.2) as well. If the medium is isotropic, then $\chi_{rr} = \chi_{zz}$, $\chi_{rz} = 0$ and Eq. (0.2) changes into the axisymmetric Laplace equation

$$T_{,rr} + T_{,zz} + T_{,r}/r = F,\ F = (-Q_0/\chi_{rr})\delta(r - r_0)\delta(z - z_0), \tag{1.1}$$

whose solution is well known from [1]:*

$$T^*(r, z; r_0, z_0) = \frac{Q_0 r_0 \mathrm{K}(\mu)}{\pi\chi_{rr}\sqrt{a + b}} = \frac{Q_0}{2\pi\chi_{rr}}\sqrt{\frac{r_0}{r}}\,Q_{-1/2}\left(\frac{a}{b}\right), \tag{1.2}$$

where $\mathrm{K}(\mu) = \int\limits_0^{\pi/2}(1 - \mu^2\sin^2\theta)^{-1/2}\,d\theta$ is the total normal elliptical Legendre integral of the first kind; $Q_{-1/2}$ is the Legendre function of the second kind;

$$a = r^2 + r_0^2 + (z - z_0)^2,\ b = 2rr_0,\ \mu = \sqrt{2b/(a + b)}. \tag{1.3}$$

Let us bring the axisymmetric steady equation to the form of (1.1), for which purpose we will change from the variables (r, z) to the variables (r, ζ):

$$\zeta = (z - \chi_1 r)/\xi,\ \chi_1 = \chi_{rz}/\chi_{rr},\ \xi = \sqrt{\chi_{zz}/\chi_{rr} - (\chi_{rz}/\chi_{rr})^2}, \tag{1.4}$$

and after substitution into (0.2) we derive the axisymmetric Laplace equation (1.1) for r, ζ relative to T^0 with the right-hand side

$$F^0(r, \zeta; r_0, \zeta_0) = (-Q_0/\chi_{rr})\,\delta(r - r_0)\,\delta\,[\xi(\zeta - \zeta_0) + \chi_1(r - r_0)].$$

Using (1.2) as the nucleus of the volume potential with density F^0, we have the sought fundamental solution of the axisymmetric steady equation (0.2)

$$T(r, z; r_0, z_0) = \frac{Q_0 r_0 \mathrm{K}(\mu^0)}{\pi\xi\chi_{rr}\sqrt{a^0 + b}} = \frac{Q_0}{2\pi\xi\chi_{rr}}\sqrt{\frac{r_0}{r}}\,Q_{-1/2}\left(\frac{a^0}{b}\right); \tag{1.5}$$

$$a^0 = r^2 + r_0^2 + \zeta_\Delta^2,\ \mu^0 = \sqrt{2b/(a^0 + b)}. \tag{1.6}$$

Here and below, $r_\Delta = r - r_0$, $z_\Delta = z - z_0$, $\zeta_\Delta = \zeta - \zeta_0 = (z_\Delta - \chi_1 r_\Delta)/\xi$; $R = \sqrt{r_\Delta^2 + \zeta_\Delta^2}$. When we take into consideration the properties of the Legendre functions of the second kind and their relationship to the elliptical integrals [6, 7], from the second of the equalities in (1.5) we can derive expressions for the gradient of the fundamental solution

$$\nabla T = T_{,r}(r, z; r_0, z_0)\,\mathbf{e}_r + T_{,z}(r, z; r_0, z_0)\,\mathbf{e}_z =$$

*Let us note that in [1] for the elliptical integral it is apparently the definition of $K(m) = \int\limits_0^{\pi/2}(1 - m\sin^2\theta)^{-1/2}\,d\theta$ that is used and therefore in an expression such as (1.2) it is the parameter $m = \mu^2$ that plays a role as opposed to μ. There is a printing error in the expression for the argument of the Legendre function.

$$= \frac{-Q_0 r_0}{\pi\xi\chi_{rr}\sqrt{a^0+b}} \left\{ \frac{1}{2r}\left[\left(r^2 - r_0^2 - \zeta_\Delta^2 - 2\frac{\chi_1}{\xi} r\zeta_\Delta\right)\frac{\mathbf{E}(\mu^0)}{R^2} + \mathbf{K}(\mu^0)\right]\mathbf{e}_r + \right.$$

$$\left. + \frac{\zeta_\Delta}{\xi R^2}\mathbf{E}(\mu^0)\mathbf{e}_z \right\} = \frac{Q_0}{2\pi\xi\chi_{rr}}\left(\frac{r_0}{r}\right)^{1/2}\left\{\frac{1}{(a^{02}-b^2)}\left[-a^0 Q_{-1/2}\left(\frac{a^0}{b}\right) + \right.\right.$$

$$\left.\left. + bQ_{1/2}\left(\frac{a^0}{b}\right)\right]\left[\frac{1}{2r}\left(r^2 - r_0^2 - \zeta_\Delta^2 - 2\frac{\chi_1}{\xi} r\zeta_\Delta\right)\mathbf{e}_r + \frac{\zeta_\Delta}{\xi}\mathbf{e}_z\right] - \frac{1}{2r}Q_{1/2}\left(\frac{a_0}{b}\right)\mathbf{e}_r\right\}$$

$\left(\mathbf{E}(\mu) = \int_0^{\pi/2} (1 - \mu^2 \sin^2\theta)^{1/2}\, d\theta\right.$ is the total normal elliptical Legendre integral of the second kind).

Let us also dwell in some detail on the limit properties of the fundamental solution as $r/r_0 \to 0$, $r_0/R \to 0$, $R/r_0 \to 0$:

$$\frac{r}{r_0} \to 0 \Rightarrow T \to \frac{Q_0}{2\xi\chi_{rr}}\left(1 + \frac{\zeta_\Delta^2}{r_0^2}\right)^{-1/2}, \quad \nabla T \to \frac{Q_0}{2\xi^2\chi_{rr}}\frac{\zeta_\Delta}{r_0^2}\left(1 + \frac{\zeta_\Delta^2}{r_0^2}\right)^{-3/2}(\chi_1\mathbf{e}_r - \mathbf{e}_z);$$

$$\frac{r_0}{R} \to 0 \Rightarrow T \to \frac{Q_0}{2\xi\chi_{rr}}\frac{r_0}{R}, \quad \nabla T \to \frac{-Q_0}{2\xi^3\chi_{rr}^2 R^2}\frac{r_0}{R}\left[(\chi_{zz}r_\Delta - \chi_{rz}z_\Delta)\mathbf{e}_r - \right. \tag{1.7}$$

$$\left. - (\chi_{rz}r_\Delta - \chi_{rr}z_\Delta)\mathbf{e}_z\right].$$

These limit values have been obtained from the representation of the solution in terms of elliptical integrals. For their derivation as $R/r_0 \to 0$ we will use the representations of the solutions in terms of the Legendre functions and we will take into consideration that $a^0/b = 1 + (R/r_0)^2/2 + O(R^3/r_0^3)$, while for small η [6]

$$Q_\nu(1+\eta) = -\frac{1}{2}\ln\left(\frac{\eta}{2}\right) - \gamma - \psi(1+\nu) + O(\eta\ln\eta)$$

(ψ is the logarithmic derivative of the Γ-function and γ is the Euler–Mascheroni constant). Then, if we denote $c_1 = \gamma + \psi(1/2)$, we have

$$\frac{R}{r_0} \to 0 \Rightarrow T \to \frac{-Q_0}{2\pi\xi\chi_{rr}}[\ln R - \ln(2r_0) + c_1] + O\left[\left(\frac{R}{r_0}\right)^2 \ln\frac{R}{r_0}\right],$$

$$\nabla T \to \frac{-Q_0}{4\xi^3\chi_{rr}^2 R^2}[(\chi_{zz}r_\Delta - \chi_{rz}z_\Delta)\mathbf{e}_r - (\chi_{rz}r_\Delta - \chi_{rr}z_\Delta)\mathbf{e}_z] + \frac{1}{r_0}O\left[\frac{R}{r_0}\ln\left(\frac{R}{r_0}\right)\right]. \tag{1.8}$$

Not only do relationships (1.8) yield the principal terms of the fundamental solution in the case of a small distance between the source and the point of observation, but they also make it possible to derive a fundamental solution in the plane problem with rectilinear anisotropy, into which the cylindrical anisotropy as r, $r_0 \to \infty$ degenerates. However, it should be kept in mind that in the axisymmetric problem the value of the function T, as follows from (1.7), is calculated from its value at infinity, whereas in the plane problems the fundamental solution at infinity may be unbounded. Therefore, we will calculate T from some fixed point r_1, z_1:

$$T^{(p)}(r_\Delta, z; 0, z_0) = \lim_{r_0\to\infty}[T(r_0 + r_\Delta, z; r_0, z_0) - T(r_0 + r_{1\Delta}, z_1; r_0, z_0)].$$

When we take (1.8) into consideration the fundamental solution for $T^{(p)}$ in the plane steady heat-conduction problem for a rectilinear anisotropic medium has the form

$$T^{(p)}(r_\Delta, z; 0, z_0) = -Q_0(2\pi\xi\chi_{rr})^{-1}\ln R + c_0,$$

where the constant $c_0 = Q_0(\pi\xi\chi_{rr})^{-1}\ln(r_{1\Delta}^2 + \zeta_{1\Delta}^2)$ and it can be dropped. Representations of $T_{,r}^{(p)}$ and $T_{,z}^{(p)}$ are given by the corresponding relationships (1.8), provided that we drop the last terms in these.

2. The Axisymmetric Fundamental Solution of the Nonsteady Heat-Conduction Equation. We will seek the solution of Eq. (0.2) with the right-hand side $Q(r, \theta; z, t) = Q_0\delta(r - r_0)\delta(z - z_0)\delta(t - t_0)$. Since the right-hand side is independent of θ, this dependence also disappears in (0.2).

For an isotropic medium and analogous equation has the form

$$T_{,rr} + T_{,zz} + T_{,r}/r - (c/\chi_{rr})T_{,t} = F,\ F = (-Q_0/\chi_{rr})\,\delta(r - r_0)\,\delta(z - z_0)\,\delta(t - t_0). \quad (2.1)$$

We know the solution for (2.1) from [1]:

$$T^*(r, z, t; r_0, z_0, t_0) = \frac{2\pi Q_0 r_0}{c\,(4\pi k t_\Delta)^{3/2}} \exp\left(-\frac{a}{4kt_\Delta}\right) I_0\left(\frac{b}{4kt_\Delta}\right) H(t_\Delta). \quad (2.2)$$

Here $t_\Delta = t - t_0$, $k = \chi_{rr}/c$; a and b are represented by relationships (1.3); $H(\tau)$ is the Heaviside function; I_0 is a modified Bessel function of the first kind, of zeroth order.

As was done in Sec. 1, replacing the variables from (1.4) in Eq. (0.2), we arrive at (2.1), from which, using (2.2), we write the sought fundamental solution of the nonsteady axisymmetric equation (0.2)

$$\overline{T}(r, z, t; r_0, z_0, t_0) = \frac{2\pi Q_0 r_0}{c\xi\,(4\pi k t_\Delta)^{3/2}} \exp\left(-\frac{a^0}{4kt_\Delta}\right) I_0\left(\frac{b}{4kt_\Delta}\right) H(t_\Delta)$$

[a_0 is defined by relationship (1.6)]. After differentiation we derive the gradient of the fundamental solution

$$\nabla\overline{T} = \overline{T}_{,r}\mathbf{e}_r + \overline{T}_{,z}\mathbf{e}_z = \frac{4Q_0 r_0}{c\xi\sqrt{\pi}\,(4kt_\Delta)^{5/2}} \exp\left(-\frac{a^0}{4kt_\Delta}\right) \times$$
$$\times \left\{\left[-\left(r - \zeta_\Delta\frac{\chi_1}{\xi}\right) I_0\left(\frac{b}{4kt_\Delta}\right) + r_0 I_1\left(\frac{b}{4kt_\Delta}\right)\right]\mathbf{e}_r - \frac{\zeta_\Delta}{\xi} I_0\left(\frac{b}{4kt_\Delta}\right)\mathbf{e}_z\right\} H(t_\Delta).$$

For the limits as $rr_0/(kt_\Delta) \to 0$, $kt_\Delta/R^2 \to 0$, $kt_\Delta/(rr_0)$, $R/r_0 \to 0$ we have

$$\begin{aligned}
&\frac{rr_0}{kt_\Delta} \to 0 \Rightarrow \overline{T} \to \frac{2\pi Q_0 r_0}{c\xi\,(4\pi k t_\Delta)^{3/2}} \exp\left(\frac{-R^2}{4kt_\Delta}\right) H(t_\Delta),\\
&\nabla\overline{T} \to \frac{4Q_0 r_0 \zeta_\Delta}{c\xi^2\sqrt{\pi}\,(4kt_\Delta)^{5/2}} \exp\left(-\frac{R^2}{4kt_\Delta}\right)[\chi_1\mathbf{e}_r - \mathbf{e}_z]\,H(t_\Delta),\\
&\frac{kt_\Delta}{R^2} \to 0 \Rightarrow T = O\left(\exp\left(-\frac{R^2}{4kt_\Delta}\right)\right),\quad \nabla T = O\left(\exp\left(-\frac{R^2}{4kt_\Delta}\right)\right);\\
&\frac{kt_\Delta}{rr_0} \to 0,\quad \frac{R}{r_0} \to 0 \Rightarrow T = \frac{Q_0}{c\xi 4\pi k t_\Delta}\left[\exp\left(-\frac{R^2}{4kt_\Delta}\right) + O\left(\frac{kt_\Delta}{rr_0}\right)\right],\\
&\nabla T \to \frac{2Q_0}{\pi c\xi^2\chi_{rr}\,(4kt_\Delta)^2} \exp\left(-\frac{R^2}{4kt_\Delta}\right)\Big\{(-\chi_{zz}r_\Delta + \chi_{rz}z_\Delta)\,\mathbf{e}_r +\\
&\qquad + (\chi_{rz}r_\Delta - \chi_{rr}z_\Delta)\,\mathbf{e}_z + \chi_{rr}RO\left(\frac{kt_\Delta}{rr_0}\right)\Big\}.
\end{aligned} \quad (2.3)$$

As in the case of nonsteady heat conduction, relationships (2.3) yield not only the asymptote of the fundamental solution for limited times and at a limited distance between the source and the point of observation, but also the fundamental solution of the plane nonsteady heat-conduction problem for a rectilinear anisotropic medium, provided that the last terms in these formulas are dropped.

LITERATURE CITED

1. K. Brebbia, G. Telles, and L. Wrobel, The Methods of Boundary Elements [Russian translation], Mir, Moscow (1987).
2. C. Patterson and M. A. Sheikh, "A modified Trefftz method for three-dimensional elasticity," in: Boundary Elements: Proceedings, 5th Int. Conf., Hiroshima, 1983, Springer, Berlin (1983).
3. I. M. Lifshits and L. N. Rosentsveig, "The construction of the Green's tensor for the basic equation of the theory of elasticity in the case of an unbounded elastic anisotropic medium," Zh. Éksp. Teor. Fiz., 17, No. 9 (1947).

4. S. E. Mikhailov and A. E. Osokin, "Construction of the fundamental solution for an anisotropic aging hereditary-type medium," Dokl. Akad. Nauk SSSR, 274, No. 2 (1984).
5. S. E. Mikhailov and A. E. Osokin, "Construction of the fundamental solutions for spatial and plane problems in the case of an anisotropic hereditary-elastic aging medium," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 5 (1986).
6. H. Bateman and A. Erdane, Higher Transcendental Functions (the Hypergeometric Function, the Legendre Function) [Russian translation], Fizmatgiz, Moscow (1973).
7. M. Abramovitz and M. Stigan (eds.), Handbook of Special Functions [Russian translation], Nauka, Moscow (1979).

CONTACT PROBLEM IN THE THEORY OF ELASTICITY FOR NARROW AREAS, WITH WEAR TAKEN INTO CONSIDERATION

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UDC 539.3

1. We examine the spatial steady contact problem for the theory of elasticity in the presence of wear. Let a body 1 slide relative to body 2; let there be no wear in this case, and let the linear wear j for body 2 be proportional to the work of the force of friction [1]

$$j = K^* \mu l p_1,$$

where p_1 is the pressure; μ and K^* are the coefficients of friction and proportionality between the work of the force of friction and the volume of material removed; ℓ represents the friction path.

Let us choose an affine system of coordinates $Ox_1y_1z_1$, connected to the contact (the Oz_1 axis is perpendicular to the contact and directed toward body 1), so that $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ exhibits unit length, and the angle between $\mathbf{e}_x$ and $\mathbf{e}_y$ is equal to β (see Fig. 1).

Let the field of the vector for the sliding velocity be uniformly plane-parallel: $\mathbf{V} = -v\mathbf{e}_y$, the area of contact $G_1 = \{(x_1, y_1): x_1^- \le x_1 \le x_1^+, y_1^-(x_1) \le y_1 \le y_1^+(x_1)\}$ [$y_1^\pm(x_1)$ are continuous functions]. The shape of the bodies and of the contact is independent of time. This hypothesis is valid, for example, in the following cases: a) 2 represents the half space; b) 1 is the rocking body and 2 is the bearing ring.

The equation from the theory of elasticity, with wear taken into consideration, has the form

$$\theta \iint_{G_1} \frac{p_1(\xi_1, \eta_1)\, d\xi_1\, d\eta_1 \sin\beta}{r(\xi_1, \eta_1, x_1, y_1)} = w_1(x_1, y_1) - K^*\mu \int_{y}^{y_1^+(x_1)} p_1(x_1, \eta_1)\, d\eta_1. \qquad (1.1)$$

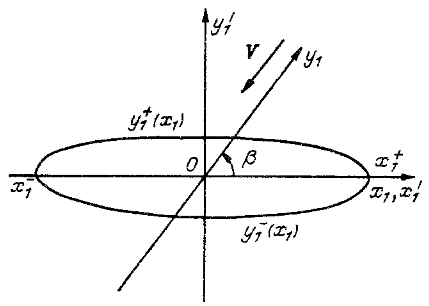

Fig. 1

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